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DISPERSION OF NONLINEAR WAVES IN A ROD*

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Equations from the linear theory of transverse oscillations of a rod have solutions of the form $e^{i\theta}, \theta = \lambda\xi - \omega t$ if the wave number λ and frequency ω are linked by the dispersion relation $f(\lambda, \omega) = 0$. In the present paper, a nonlinear generalization of such solutions and a corresponding dispersion relation is constructed for plane oscillations of a inextensible rod. The basic nonlinear effects, like previously studied cases /1/, are seen in the fact that the harmonic oscillation $e^{i\theta}$ is transformed into an oscillation of the form $\psi(\theta), \theta = \theta(\xi, t)$ where $\psi(\theta)$ is a periodic function of θ, further in the dispersion relation a dependence on the amplitude of the oscillations arises while the derivatives θ_ξ and $-\theta_t$ serve as the wave number and frequencies. The nature of the problem is related to the introduction of a particular "effective" rod characterizing slow mean motion on which rapid oscillations with wave number θ_ξ and frequency θ_t are superimposed. The energy of the effective rod is computed and equations are constructed that describe its evolution. The dispersion relation arises as an equilibrium condition imposed on the effective rod.

The main purpose of the present paper is to explain whether the Whitham method in the theory of nonlinear waves /1/ is a particular case of the general method of studying functionals that depend upon a small parameter proposed in /2/. A positive answer to this question is given.

1. Formulation of problem. We consider in a Cartesian coordinate system x^1, x^2 an infinite rod that coincides with the axis x^1 in its undeformed state. The axes x^2, x^3 are the axes of inertia of the cross-section. The length of an arc along the rod is denoted by ξ. The motion of the rod is described by the displacements $u^i(\xi, t), i = 1, 2$. The rod is assumed to be incompressible:

$$u_{1\xi} + \frac{1}{2} u_{i\xi} u^i_\xi = 0 \tag{1.1}$$

Differentiation with respect to ξ and t are denoted by subscripts "ξ" and "t". The Lagrangian of the rod referred to half the area of the cross-section $|S|$ and the Young's modulus E has the form

$$\Lambda = \bar{h}^2 u^i_{\xi\xi} u_{i\xi\xi} - \rho E^{-1} u^i_t u_{it}, \quad \bar{h} = (I/|S|)^{1/2} \tag{1.2}$$

Here I is the moment of inertia of the cross-section and ρ is the volume density of the mass. No constraints are imposed on the amplitude of the displacements. The oscillations of the rod are described by the nonlinear theory, and the nonlinear effects occur only in equation (1.1).

We will find solutions of the form $u^i = f^i(\theta, \xi, t)$, where f^i are periodic (with minimal period 2π) functions of θ, θ is a function of ξ and t, and the characteristic scales L and T of variation of the functions $f^i(\theta, \xi, t)$ (at fixed θ) and θ_ξ, θ_t with respect to ξ and t are much greater than the characteristic scales l, τ of variation of the function $\theta(\xi, t)$ with respect to ξ and t. It is assumed that the functions f^i have periodic first derivatives with respect to θ: $[f_\theta^i]_{-\pi}^{\pi} = 0$ ($[A]_{-\pi}^{\pi}$ is the difference in the values of the function $A(\theta)$ and $\theta = \pi$ and $\theta = -\pi$ (while the subscript "θ" denotes differentiation with respect to θ).

We let $\langle\cdot\rangle$ denote the operation of averaging with respect to θ.

$$\langle A(\theta)\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\theta)\, d\theta$$

We introduce an effective rod whose axis is determined by the radius-vector with components $v^i(\xi, t) = \langle f^i(\theta, \xi, t)\rangle$. Then the components of the displacement vector of the rod may be

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$$u^i = v^i(\xi, t) + \psi^i(\theta, \xi, t), \quad \theta = \theta(\xi, t) \tag{1.3}$$

The functions ψ^i satisfy the constraints

$$\langle \psi^i \rangle = 0 \tag{1.4}$$

The oscillations (1.3) impose rapid micro-oscillations $\psi^i(\theta, \xi, t)$ on the mean slow oscillation $v^i(\xi, t)$. It is necessary to find the rapidly varying part of the oscillations ψ^i (to a first approximation in the small parameters a/L, l/L, and τ/T, with a the amplitude of the oscillations) and to construct equations for the mean characteristics of the oscillations $v^i(\xi, t)$ and $\theta(\xi, t)$. The phase $\theta(\xi, t)$ may be considered as an additional internal degree of freedom of the effective rod.

Remark 1°. The Lagrangian (1.2) also describes the free oscillations of a plate, such that the displacements depend only on the coordinates ξ and t. In (1.2), then, we must set $\bar{h} = h/\sqrt{12}$ and replace E by $2\mu/(1-\nu)$, where h is the thickness of the plate; μ, shear modulus; and ν, the Poisson coefficient.

2°. The inextensibility condition imposed on the rod is adopted to simplify the discussion; the entire study may be repeated nearly word for word for the case of stretchable rods if it is assumed that the stretching is small by comparison with the unit of length. The assumption of inextensibility in the description of transverse oscillations with small amplitudes is of no importance, since the transverse and longitudinal oscillations are independent. In the case of high amplitudes, the error in the inextensibility condition is the smaller, the higher the oscillation amplitude. The dispersion relation in the intermediate case has recently been constructed /3/.

2. Method of solution. We establish functions $v^i(\xi, t)$ and $\theta(\xi, t)$ and find out how ψ^i depends upon v^i and θ. The problem contains small parameters l/L and τ/T, so that it is natural to apply the variational-asymptotic method /2/. In accordance with the general technique, we will substitute (1.3) in (1.2) and retain only principal terms with respect to ψ^i and the principal cross terms. We obtain the Lagrangian Λ_0, with

$$\langle \Lambda_0 \rangle = \langle \varkappa^2 \theta_\xi^2 \psi^i_{\theta\theta} \psi_{i\theta\theta} - \rho E^{-1} \theta_t^2 \psi^i_\theta \psi_{i\theta} \rangle, \quad \varkappa = \bar{h}\theta_\xi \tag{2.1}$$

By (1.1) the functions ψ^i satisfy the constraint

$$(1 + v_{1\xi} + \theta_\xi \psi_{1\theta})^2 + (v_{2\xi} + \theta_\xi \psi_{2\theta})^2 = 1 \tag{2.2}$$

The computation of ψ^i reduces to finding the fixed points of the functional (2.1) on a set of functions that satisfy the conditions

$$[\psi^i]_{-\pi}^{\pi} = 0, \quad [\psi^i_\theta]_{-\pi}^{\pi} = 0 \tag{2.3}$$

and the constraints (1.4) and (2.2). In this case v_ξ^i, θ_ξ and θ_t are considered to be constant parameters.

Let us assume that the problem has been solved. Then ψ^i will be known as functions of θ and the parameters $v_\xi^i, \theta_\xi, \theta_t$. An explicit relation ψ^i from ξ and t arises because of the relation between ξ and t and the parameters $v_\xi^i, \theta_\xi, \theta_t$. Let us consider the rod action functional in the functions (1.3), in which ψ^i is understood to refer to the solution of the problem (2.1)—(2.3), (1.4). Then the action functional becomes a functional of the slowly varying functions v^i and θ, and the corresponding Lagrangian $\bar{\Lambda}$ is given, to a first approximation, by the formula $\bar{\Lambda} = \langle \Lambda \rangle$, where Λ is considered with respect to the displacements (1.3) with known functions $\psi^i(\theta, v_\xi^i, \theta_\xi, \theta_t)$. The unknown values v^i and θ will become the fixed points of the averaging functional. Thus, the first step of the variational-asymptotic method entirely coincides with the Whitham method in this problem.

Let us now transform the problem of determining ψ^i to a more convenient form. By (2.2), there exists a function $\varphi(\theta)$, such that

$$1 + v_{1\xi} + \theta_\xi \psi_{1\theta} = \cos\varphi, \quad v_{2\xi} + \theta_\xi \psi_{2\theta} = \sin\varphi \tag{2.4}$$

This function has the meaning of the angle between the tangent to the deformed axis of the rod and the axis x^1. In place of ψ^i, we will find the function $\varphi(\theta)$. The function ψ is found from (2.4) using the already known function $\varphi(\theta)$.

The periodicity condition on ψ^i is equivalent to the following constraints on $\varphi(\theta)$:

$$1 + v_{1\xi} = \langle \cos\varphi \rangle, \quad v_{2\xi} = \langle \sin\varphi \rangle \tag{2.5}$$

The periodicity condition ψ_θ^i states that, according to (2.4), $[\varphi]_{-\pi}^{\pi} = 2\pi s$ where s is an integer. The case $|s| > 1$ is not possible, since here 2π will not be the minimal period of ψ^i. The case $[\varphi]_{-\pi}^{\pi} = -2\pi$ reduces to the case $[\varphi]_{-\pi}^{\pi} = 2\pi$ by means of the substitution $v_2, \psi_2, \varphi \to -v_2, -\psi_2, -\varphi$. Thus

$$[\varphi]_{-\pi}^{\pi} = 0 \quad \text{and} \quad [\varphi]_{-\pi}^{\pi} = 2\pi \tag{2.6}$$

After eliminating from (2.1) derivatives of ψ^i using the relations in (2.4), we find that the unknown function $\varphi(\theta)$ is a fixed point of the functional

$$\langle\Lambda_0\rangle = \langle \varkappa^2\varphi_\theta^2 - 2\alpha[(1+v_{1\xi})\cos\varphi + v_{2\xi}\sin\varphi] - 2\alpha(1+\gamma_v)\rangle \tag{2.7}$$
$$\alpha = \rho c^2/E, \quad c = \theta_t/\theta_\xi, \quad \gamma_v = v_{1\xi} + \frac{1}{2}v_{i\xi}v_\xi^i$$

under the constraints (2.5), (2.6). Here c is the phase velocity (α has the meaning of the square of the dimensionless phase velocity), and γ_v is a measure of the tension on the effective rod.

3. Computing the function $\varphi(\theta)$. Let us depart from the constraints (2.5), adding to the functional the expression

$$2\alpha\lambda(\cos\varphi - 1 - v_{1\xi}) + 2\alpha\mu(\sin\varphi - v_{2\xi})$$

where λ and μ are Lagrangian multipliers. In place of the two parameters λ and μ, it is more convenient to use the quantities m and φ_*, which are determined by the relations

$$m^2 = \alpha[(1+v_{1\xi}+\lambda)^2 + (v_{2\xi}+\mu)^2]^{1/2}$$
$$\cos\varphi_* = \alpha(1+v_{1\xi}+\lambda)/m^2, \quad \sin\varphi_* = \alpha(v_{2\xi}+\mu)/m^2$$

We also introduce the angle of "mean" inclination φ_v by means of the equalities

$$1 + v_{1\xi} = \sqrt{1+2\gamma_v}\cos\varphi_v, \quad v_{2\xi} = \sqrt{1+2\gamma_v}\sin\varphi_v$$

Then the functional (2.7) may be rewritten in the form

$$\langle\Lambda_0\rangle = \langle x^2\varphi_\theta^2 + 2m^2\cos\overline{\varphi}\rangle - 2m^2\sqrt{1+2\gamma_v}\cos(\varphi_* - \varphi_v) + 2\alpha\gamma_v \quad (\overline{\varphi} = \varphi - \varphi_*) \tag{3.1}$$

We now pass from the unknown quantities φ, λ, μ to the quantities $\overline{\varphi}, m, \varphi_*$. The fixed points with respect to φ_* are easily found:

$$\varphi_* = \varphi_v + (0 \text{ or } \pi) \tag{3.2}$$

Here it is taken into account that displacements along φ_* by the quantity $2\pi s$ (with s an integer) will not be substantial.

In place of γ_v, it is convenient to introduce a new mean tension measure $\overline{\gamma}$ determined by the equalities

$$\gamma_v = \overline{\gamma} + \frac{1}{2}\overline{\gamma}^2, \quad \overline{\gamma} = -1 \pm \sqrt{1+2\gamma_v} \tag{3.3}$$

with the plus sign in (3.3) used if $\varphi_* = \varphi_v$ and the minus sign if $\varphi_* = \varphi_v + \pi$. The quantity γ_v varies over the range $[-1, 0]$, and the corresponding values of $\overline{\gamma}$ lie in the segment $[-2, 0]$.

Finding $\overline{\varphi}$ and m reduces to computing the fixed points of the functional

$$\langle\Lambda_0\rangle = \langle\varkappa^2\overline{\varphi}_\theta^2 + 2m^2(\cos\overline{\varphi} - 1) + 2(\alpha - m^2)\overline{\gamma} + \alpha\overline{\gamma}^2\rangle \tag{3.4}$$

The fixed points satisfy the equations

$$\varkappa^2\overline{\varphi}_{\theta\theta} + m^2\sin\overline{\varphi} = 0 \tag{3.5}$$
$$[\overline{\varphi}]_{-\pi}^{\pi} = 0 \text{ or } 2\pi, \quad [\overline{\varphi}_\theta]_{-\pi}^{\pi} = 0, \quad \langle\cos\overline{\varphi} - 1\rangle = \overline{\gamma} \tag{3.6}$$

If $\overline{\varphi}(\theta)$ is a fixed point then $\pm\overline{\varphi}(0 + \text{const}) + 2\pi s$ (s is an integer) is also a fixed point. The corresponding arbitrariness is of no importance when determining the deformed state. Consequently, it may be assumed that $|\overline{\varphi}(-\pi)| \leqslant 2\pi$. Thus from the phase picture of the equation (3.5) it is clear that the point $\overline{\varphi}(0)$ always belongs to the solution. Also using the fact that $\varphi_\theta \neq 0$ if $\overline{\varphi} = 0$, we find the additional constraints

$$\overline{\varphi}(0) = 0, \quad \overline{\varphi}_\theta(0) < 0 \tag{3.7}$$

The "physical pendulum" equation (3.5) has (taking into account (3.7)) a family of solutions that depend upon the parameter k /4/:

$$\sin\frac{\overline{\varphi}}{2} = -k\,\mathrm{sn}\left(\frac{m\theta}{\varkappa}, k\right), \quad k < 1 \tag{3.8}$$
$$\sin\frac{\overline{\varphi}}{2} = -\mathrm{sn}\left(\frac{mk\theta}{\varkappa}, \frac{1}{k}\right), \quad k > 1$$
$$\sin\frac{\overline{\varphi}}{2} = -\mathrm{th}\left(\frac{m\theta}{\varkappa}\right), \quad k = 1$$

Let us consider the cases (3.8) separately.

Case $k<1$. From the phase picture of equation (3.5), it is clear that the case $[\varphi]_{-\pi}^{\pi}=0$ is realized. The form of the oscillations is found from (2.4):

$$\theta_\xi\psi_2=-v_{2\xi}\theta+\frac{2k\varkappa}{m}\operatorname{cn}\frac{m\theta}{\varkappa}\cos\varphi_*+\left(\theta-\frac{2\varkappa}{m}E\left(am\frac{m\theta}{\varkappa},\ k\right)\right)\sin\varphi_*+\text{const} \tag{3.9}$$

The additive constant in (3.9) is determined from the constraint (1.4) and is insignificant henceforth.

If $v_{2\xi}=0$, $\cos\varphi_*=1$ the oscillations have the form (a is the amplitude)

$$\psi_2=a\operatorname{cn}\left(\frac{m\theta}{\varkappa},k\right),\quad a=\frac{2k\varkappa}{m\theta_\xi} \tag{3.10}$$

At low k, the bar in the deformed state has the form of a sinusoidal curve. As k increases, the shape of the bar increasingly deviates from a sinusoidal curve. When $k=1/\sqrt{2}$, there arise, by (3.8), points at which the tangent to the bar is perpendicular to the effective bar axis $(\overline{\varphi}=\pi/2)$ and when $k>1/\sqrt{2}$, the projection of the bar on the effective axis is no longer unique, further at values of k close to unity the curve crosses itself.

Conditions (3.6) impose definite constraints on the parameters $m,\varkappa$ and k. The constraints are obtained more conveniently from the stationarity condition imposed on the functional (3.4) relative to the amplitude parameter k (as has been previously suggested by Whitham /1/, rather than from (3.6); in our problem we also required a stationarity condition with respect to m.

Let us compute the functional (3.4). The fixed points (3.4) satisfy the equation

$$\varkappa^2\overline{\varphi}_\theta^2+2m^2(1-\cos\overline{\varphi})=\varkappa^2\overline{\varphi}_\theta^2+4m^2\sin^2\frac{\overline{\varphi}}{2}=\text{const} \tag{3.11}$$

The constant in (3.11) is taken in the form $4m^2k^2$. Then

$$\overline{\varphi}_\theta=-\frac{2m}{\varkappa}\sqrt{k^2-\sin^2\frac{\overline{\varphi}}{2}} \tag{3.12}$$

Let us rewrite (3.4) in the form

$$\langle\Lambda_0\rangle=\frac{\varkappa^2}{\pi}\oint\overline{\varphi}_\theta\,d\overline{\varphi}-4m^2k^2+2(\alpha-m^2)\overline{\gamma}+\alpha\overline{\gamma}^2 \tag{3.13}$$

The integral in (3.13) is taken over the range of variation of $\overline{\varphi}$ for the entire period, and the quantity $\overline{\varphi}_\theta$ is determined by (3.12). To compute the integral in (3.13), we make the substitution $\overline{\varphi}(\theta)\to\chi(\theta)$, $\sin\overline{\varphi}/2=-k\operatorname{sn}(\chi,k)$. The quantity χ varies in the range $|\chi|\leqslant K(k)$. We find

$$\frac{\varkappa^2}{\pi}\oint\overline{\varphi}_\theta\,d\overline{\varphi}=\frac{4m\varkappa k^2}{\pi}\oint\operatorname{cn}^2\chi\,d\chi=\frac{16m\varkappa}{\pi}(E(k)-(1-k^2)K(k))$$

where E and K are complete elliptic integrals. Thus

$$\langle\Lambda_0\rangle=\frac{16m\varkappa}{\pi}(E-(1-k^2)K)-4m^2k^2+2(\alpha-m^2)\overline{\gamma}+\alpha\overline{\gamma}^2 \tag{3.14}$$

The fixed points for the functional (3.14) with respect to m and k satisfy the equations

$$m=\frac{2}{\pi}K(k)\varkappa,\quad\frac{4\varkappa}{\pi m}[E(k)-(1-k^2)K(k)]-2k^2=\overline{\gamma} \tag{3.15}$$

Eliminating m from (3.15) leads to a relation between the amplitude parameter k and the tensile measure of the effective rod $\overline{\gamma}$:

$$2(E(k)K^{-1}(k)-1)=\overline{\gamma} \tag{3.16}$$

The function $2\ (EK^{-1}-1)$ decreases monotonically from 0 down to -2 at values of k that increase from 0 to 1. The quantity $\overline{\gamma}$ varies, when $0\leqslant k\leqslant 1/\sqrt{2}$ in the range from 0 to $\gamma_*=-0.543$.

Because of the one-to-one relation between the amplitude parameter k and $\overline{\gamma}$, we may consider γ as a measure of the amplitude of the oscillations.

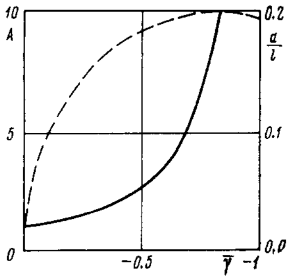

Fig.1

The true amplitude a, refers to a new wavelength $l = 2\pi/\theta_\xi$, as given by (3.10), by the equality

$$a/l = {}^1/_2 k/K(k)$$

Replacing k by $\overline{\gamma}$ from (3.16) in the latter equality, we arrive at a relation between the dimensionless amplitude and $\overline{\gamma}$, a relation which is plotted in the accompanying figure by the broken line. The maximal value of the dimensionless amplitude a/l is reached at $k = 0.83$, $\overline{\gamma} = -0.79$, and is equal to 0.2.

Case $k > 1$. From the phase picture of equation (3.5), it is clear that when $k > 1$, the case $[\varphi]_{-\pi}^{\pi} = 2\pi$ is realized. Therefore, the oscillations constitute a set of loops moving along the rod. As in the above,

$$\langle\Lambda_0\rangle = \frac{8mk\varkappa}{\pi} E\left(\frac{1}{k}\right) - 4m^2k^2 + 2(\alpha - m^2)\overline{\gamma} + \alpha\overline{\gamma}^2 \tag{3.17}$$

The fixed points with respect to m and k satisfy the equations

$$m = \frac{\varkappa K}{\pi k}, \quad \frac{2\varkappa k}{\pi m} E\left(\frac{1}{k}\right) - 2k^2 = \overline{\gamma}$$

After eliminating m, we obtain a relation between k and $\overline{\gamma}$:

$$2k^2\left[E\left(\frac{1}{k}\right)K^{-1}\left(\frac{1}{k}\right) - 1\right] = \overline{\gamma} \tag{3.18}$$

The left side of (3.18) increases monotonically from -2 to -1 as k increases from 1 to ∞.

Case $k = 1$. Such oscillations do not satisfy the periodicity conditions at any values of m. Therefore, when considering these oscillations the statement of the problem must be altered somewhat. We set

$$v^i = 0, \quad -\infty \leqslant \theta \leqslant +\infty, \quad \theta = \theta_\xi\xi + \theta_t t, \quad \theta_\xi, \theta_t = \text{const}$$

and consider the function ψ^i in (1.3) to be particular, not necessarily periodic functions of θ. Repeating the arguments presented above in our determination of $\overline{\varphi} = \varphi$ word for word, we obtain the functional (3.4) with $m^2 = -\alpha$. Therefore, we have oscillations determined by the last relation in (3.8) with the form of the oscillations

$$\psi_2 = a\,\mathrm{ch}^{-2}\frac{\xi - ct}{a}, \quad a = 2\alpha^{-1/2}\hbar \tag{3.19}$$

The angle $\overline{\varphi}$ varies as θ increases from $-\infty$ to $+\infty$ in the range $[0,2\pi]$. Motion of an isolated loop (soliton) along the rod corresponds to the solution. The speed of the soliton c and the amplitude a are related, by (3.19) by the formula

$$ac = 2\sqrt{EI/\rho|S|}$$

Note that unlike waves on a water surface /1/, the speed of the soliton is inversely proportional to the amplitude.

4. Effective Lagrangians.

The value $\langle\Lambda_0\rangle$ in the fixed points is given by the functions $\overline{\gamma}, \alpha$ and $\varkappa$. We denote $F(\overline{\gamma}, \alpha, \varkappa)$. In the case of average motion, the Lagrangian (which is obtained as a result of substituting (1.3) in (1.2), discarding small terms, and averaging with the respect to θ) is related to $F(\overline{\gamma}, \alpha, \varkappa)$ by means of the formula

$$\overline{\Lambda} = F(\overline{\gamma}, \alpha, \varkappa) + \hbar^2 v^i_{\xi\xi} v_{i\xi\xi} - \rho E^{-1} v^i_t v_{it} \tag{4.1}$$

The function F is determined by the expressions

$$k < 1: F = 2\varkappa^2\left(\frac{2}{\pi}K(k)\right)^2(\overline{\gamma} + 2k^2) + \alpha\overline{\gamma}^2 + 2\alpha\overline{\gamma} \tag{4.2}$$

$$k > 1: F = 2\varkappa^2\left(\frac{K(1/k)}{\pi k}\right)^2\overline{\gamma} + \alpha\overline{\gamma}^2 + 2\alpha\overline{\gamma} \tag{4.3}$$

Here it is assumed that in (4.2), k is a function of $\overline{\gamma}$ and determined from equation (3.16), while in (4.3), it is determined from equation (3.17).

The slowly varying functions $v^i(\xi, t)$ and $\theta(\xi, t)$ satisfy the Euler equations for the Lagrangian (4.1):

$$\frac{\rho}{E} v_{1tt} + \bar{h}^2 v_{1\xi\xi\xi\xi} = \left(\frac{1+v_{1\xi}}{1+\bar{\gamma}} \frac{\partial F}{\partial \bar{\gamma}}\right)_\xi \tag{4.4}$$
$$\frac{\rho}{E} v_{2tt} + \bar{h}^2 v_{2\xi\xi\xi\xi} = \left(\frac{v_{2\xi}}{1+\bar{\gamma}} \frac{\partial F}{\partial \gamma}\right)_\xi$$
$$\left(\frac{2\alpha}{\theta_t} \frac{\partial F}{\partial \alpha}\right)_t = \left(\frac{2\alpha}{\theta_\xi} \frac{\partial F}{\partial \alpha} - \bar{h} \frac{\partial F}{\partial \varkappa}\right)_\xi$$

5. Free oscillations and the dispersion equation. Oscillations such that $\bar{\gamma}$, α, $\varkappa$ are constant satisfy the equations (4.4). If the axis $\bar{\gamma}$ is subjected to elongation in such a way that the mean energy F has a minimum for this elongation, the corresponding oscillations may naturally be referred to as free oscillations. By (3.17), the free oscillations are given by

$$\alpha\bar{\gamma} = m^2 - \alpha \quad \text{or} \quad \alpha = m^2/(1+\bar{\gamma}) \tag{5.1}$$

Since $\alpha \geqslant 0$, $m^2 \geqslant 0$, the free oscillations will occur only when $\bar{\gamma} \geqslant -1$, which is possible when $k < 1$.

If the relation $m(k, \varkappa)$ is substituted in (5.1) and if k is expressed in terms of $\bar{\gamma}$, we obtain a dispersion equation, i.e., a relation between α, $\varkappa$ and $\bar{\gamma}$. From (5.1) and (3.15), we have

$$\alpha = A(\bar{\gamma})\varkappa^2, \quad A(\bar{\gamma}) = \frac{1}{1+\bar{\gamma}}\left(\frac{2}{\pi} K(k)\right)^2 \tag{5.2}$$

Here it is understood that k is expressed in terms of $\bar{\gamma}$, as given in equation (3.16). The case $\bar{\gamma} \to 0$ corresponds to waves of infinitely small amplitude. As $\bar{\gamma} \to 0$, we have $k \to 0$, $2\pi^{-1}K \to 1$, and the dispersion equation (5.2) turns into the classical relation of the linear theory $\alpha = \varkappa^2$.

Let us consider corrections of the corresponding infinitesimal order. At low $\bar{\gamma}$ (consequently, low k), the equation (3.16) assumes the form

$$\bar{\gamma} + k^2 = 0 \tag{5.3}$$

Consequently

$$A(\bar{\gamma}) \approx 1 - {}^3/_2\bar{\gamma} \tag{5.4}$$

Since $\bar{\gamma} < 0$, the phase velocity computed in light of nonlinearity is greater than the phase velocity computed in the linear theory. It is clear from (5.2) that this assertion is valid at finite $\bar{\gamma}$ as well. The relation between A and $\bar{\gamma}$ is depicted in the Fig.1 by a solid line.

The coefficient $A(\bar{\gamma})$ increases as $\bar{\gamma}$ varies from 0 to γ_* roughly by a factor of 3. As $\bar{\gamma} \to -1+0$, $A(\bar{\gamma}) \to +\infty$.

Let us find $F(\bar{\gamma}, \alpha, \varkappa)$ at low $\bar{\gamma}$. For this purpose, we must also take into account the next term in (5.3). We have $k^2 = -\bar{\gamma} - {}^1/_8\bar{\gamma}^2$, and from (4.2) we obtain

$$F = \left(\alpha - \frac{1}{2}\varkappa^2\right)\bar{\gamma}^2 + 2(\alpha - \varkappa^2)\bar{\gamma} \tag{5.5}$$

It turns out that the expression for (5.5) may be used not only at low $\bar{\gamma}$. Tabulating $2\pi^{-1}K(k)(2k^2+\bar{\gamma})$ as a function of $\bar{\gamma}$ shows that it may be approximated by the expression ${}^1/_4\bar{\gamma}^2 - \bar{\gamma}$; in the range of greatest interest $\gamma_* \leqslant \bar{\gamma} \leqslant 0$, the error will be at most 3.2%. Therefore formula (5.5) may be applied over the entire range $\gamma_* \leqslant \bar{\gamma} \leqslant 0$.

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